

An approximation theorem for non-decreasing functions on compact posets

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Abstract

In this short note we prove a theorem of the Stone-Weierstrass sort for subsets of the cone of non-decreasing continuous functions on compact partially ordered sets.

1 Introduction

The classic book [1] contains a theorem which states that given a compact set M and a separating semi-vector lattice S of continuous real-valued functions on M which contains the constants, there is one and only one way of making M a compact ordered space so that S becomes the set of all non-decreasing continuous real-valued functions on M . This theorem has been used in [2] to give a putative definition of noncommutative compact ordered sets. However, infimum and supremum turned out to be quite difficult to handle in the noncommutative setting. A different kind of density theorem was thus needed. Since a “continuous non-decreasing” functional calculus was available in the noncommutative context, it was natural to look for a theorem which would replace stability under infimum and supremum with stability under continuous non-decreasing functions.

Let us introduce some vocabulary in order to be more precise. Let S be a subset of the set $\mathcal{C}(M, \mathbb{R})$ of continuous real-valued functions on some space M , and $H : \mathbb{R} \rightarrow \mathbb{R}$ be a function. We will say that H *operates* on S is $H \circ f \in S$ for each $f \in S$.

As remarked in [3], the real version of the classical Stone-Weierstrass theorem can be rephrased in terms of operating functions.

Theorem 1.1 (*Stone-Weierstrass*) *Let S be a non-empty subset of $\mathcal{C}(X, \mathbb{R})$, with X a compact Hausdorff space. If*

1. *S is stable by sum,*
2. *the affine functions from \mathbb{R} to \mathbb{R} operate on S ,*

3. the function $t \mapsto t^2$ operates on S ,
then S is dense in $\mathcal{C}(X, \mathbb{R})$ for the uniform norm.

The second hypothesis is a way to say that S is a cone (hence a vector space thanks to first hypothesis) which contains the constant functions.

In fact it is proved in [4] that one can replace $t \mapsto t^2$ in the third hypothesis by any continuous non-affine function.

It is a theorem of this kind that we prove in this note, but in the same category (compact ordered sets and non-decreasing continuous functions) as the theorem of Nachbin stated above.

2 Preliminaries

Let M be a topological set equipped with a partial order \preceq . We let $I(M)$ denote the set of all continuous non-decreasing functions from M to \mathbb{R} , where \mathbb{R} has the natural topology and the natural ordering, which we write \leq , as usual. The elements of $I(M)$ are sometimes called continuous isotopies.

Let S be a subset of $I(M)$. We define the relation \preceq_S by

$$x \preceq_S y \iff \forall f \in S, f(x) \leq f(y) \quad (1)$$

It is obvious that \preceq_S is a preorder, which we call *the preorder generated by S* . This preorder will be a partial order relation if, and only if, S separates the points of M .

We say that S *generates* \preceq iff $\preceq_S = \preceq$. This is the case if, and only if, S satisfies

$$\forall a, b \in M, a \not\preceq b \implies \exists f \in S, f(a) > f(b) \quad (2)$$

Since $a \neq b \implies a \not\preceq b$ or $b \not\preceq a$, we see that if S generates \preceq , it necessarily separates the points of M .

Note that it is not guaranteed that for any poset there exists such an S generating the order. When there is one, then $I(M)$ itself will also generate the order. Posets with the property that $I(M)$ generates the order are called *completely separated ordered sets*. When M is compact and Hausdorff, complete separation is equivalent to the relation \preceq being closed in $M \times M$ (see [1]).

Let A be a set of functions from \mathbb{R} to \mathbb{R} . We will say that A operates on S iff

$$\forall H \in A, \forall f \in S, H \circ f \in S \quad (3)$$

3 Statement and proof of the theorem

Theorem 3.1 *Let (M, \preceq) be a compact Hausdorff partially ordered set. Let A be the set of continuous non-decreasing piecewise linear functions from \mathbb{R} to \mathbb{R} . Let S be a non empty subset of $I(M)$. If*

1. *S is stable by sum,*
2. *A operates on S ,*
3. *S generates \preceq .*

then S is dense in $I(M)$ for the uniform norm.

Before proving the theorem, a few comments are in order.

- First of all, the theorem is true but empty if M is not completely separated, since no S can satisfy the hypotheses in this case.
- The hypothesis that M is not empty is redundant if M has at least two elements, by 3.
- Finally, let us remark that 2 entails that S is in fact a convex cone which contains the constant functions.

To prove the theorem we need two lemmas.

Lemma 3.2 *Let $x, y \in M$ be such that $y \not\preceq x$. Then $\exists f_{x,y} \in S$ such that $0 \leq f_{x,y} \leq 1$, $f_{x,y}(x) = 0$ and $f_{x,y}(y) = 1$.*

Proof: Since S generates \preceq , there exists $f \in S$ such that $f(x) < f(y)$. Let $H \in A$ be such that $H(t) = 0$ for $t \leq f(x)$, H is affine on the segment $[f(x), f(y)]$, and $H(t) = 1$ for $t \geq f(y)$. Then $f_{x,y} := H \circ f$ meets the requirements of the lemma. \blacksquare

Lemma 3.3 *Let K, L be two compact subsets of M such that $\forall x \in K, \forall y \in L, y \not\preceq x$. Then $\exists f_{K,L} \in S$ such that $0 \leq f_{K,L} \leq 1$, $f = 0$ on K and $f = 1$ on L .*

Proof: For all $x \in K$ and $y \in L$, we find a $f_{x,y} \in S$ as in lemma 3.2. We fix a $y \in L$ and let x vary in K . Since $f_{x,y}$ is continuous, there exists an open neighbourhood V_x of x such that $f_{x,y}(V_x) \subset [0; 1/4[$. By compactity of K , there exists V_1, \dots, V_k corresponding to x_1, \dots, x_k such that $K \subset V_1 \cup \dots \cup V_k$.

Now we define $g_y := \frac{1}{k} \sum_i f_{x_i,y}$. We have $g_y \in S$ since S is a convex cone (see the last remark below the theorem). It is clear that $g_y(y) = 1$ and that for all $x \in K$, $0 \leq g_y(x) \leq \frac{1}{k}(k-1 + 1/4) = 1 - \frac{3}{4k} < 1$. We then choose $H \in A$ such that $H(t) = 0$ for $t \leq 1 - \frac{3}{4k}$ and $H(t) = 1$ for $t \geq 1$. We set $f_{K,y} := H \circ g_y$. We thus have $f_{K,y} \in S$, $f_{K,y} = 0$ on K and $f_{K,y}(y) = 1$.

Using the continuity of $f_{K,y}$, we find an open neighbourhood W_y of y such that $f_{K,y}(W_y) \subset [3/4; 1]$. Since we can do this for every $y \in$

L , and since L is compact, we can find functions f_{K,y_j} , $j = 1..l$, and open sets W_1, \dots, W_l of the above kind such that $L \subset W_1 \cup \dots \cup W_l$. We then define $g = \frac{1}{l} \sum_j f_{K,y_j}$. We have $g \in S$, and $g(K) = \{0\}$. Moreover, for all $z \in L$, $1 \geq g(z) \geq \frac{3}{4l} > 0$. We then choose a function $G \in A$ such that $G(t) = 1$ for $t \geq 3/4l$ and $G(t) = 0$ for $t \leq 0$. Now the function $f_{K,L} := G \circ g$ has the desired properties. \P

We can now prove the theorem.

Proof: Let $f \in I(M)$. We will show that, for all $n \in \mathbb{N}^*$ there exists $F \in S$ such that $\|f - F\|_\infty \leq \frac{1}{n}$.

If f is constant then the result is obvious. Else, let m be the infimum of f and M be its supremum. Let $\tilde{f} = \frac{1}{M-m}(f - m.1)$. Using the fact that S is a convex cone, we can work with \tilde{f} instead of f . Hence, we can suppose that $f(M) = [0; 1]$ without loss of generality.

We set $K_i = f^{-1}([0; \frac{i}{n}])$, and $L_i = f^{-1}([\frac{i+1}{n}; 1])$ for each $i \in \{0; \dots; n-1\}$. Since f is continuous and M is compact, the sets K_i and L_i are both closed, hence compact.

For each i we use lemma 3.3 to find $f_i \in S$ such that $f_i(K_i) = \{0\}$ and $f_i(L_i) = \{1\}$.

We then consider the function $F = \frac{1}{n} \sum_{i=0}^{n-1} f_i$. We clearly have $F \in S$.

Let $m \in M$. Suppose $\frac{j}{n} < f(m) < \frac{j+1}{n}$ for $j \in \{0; \dots; n-1\}$. We thus have $m \in K_i$ for $j < i < n$ and $m \in L_i$ for $i < j$. Hence $F(m) = \frac{1}{n} \sum_{i=0}^j f_i(m) = \frac{1}{n}(j + f_j(m)) \in [\frac{j}{n}; \frac{j+1}{n}]$. Thus $|f(m) - F(m)| \leq \frac{1}{n}$.

Now suppose $f(m) = \frac{j}{n}$, with $j \in \{0; \dots; n\}$. We have $m \in K_i$ for $i \geq j$ and $m \in L_i$ for $i < j$. Thus $F(m) = \frac{1}{n} \sum_{i=0}^{j-1} f_i(m) = \frac{j}{n}$. We see that $|f(m) - F(m)| = 0$ in this case.

Hence we have shown that $|f(m) - F(m)| \leq \frac{1}{n}$ for all $m \in M$, thus proving the theorem. \P

References

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